CRITICAL SCOUR: NEW BED PROTECTION DESIGN METHOD

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ABSTRACT: A new design philosophy for granular bed protection is given; this is referred to as the "critical scour" method. Critical scour is defined as the scour that is admissible for protection of a granular bed during its lifetime. First the traditional "critical shear" design method is discussed, and some of its shortcomings are described. These shortcomings can be overcome by application of the critical scour method. The basic philosophy of the critical scour method is given, and the two main parameters (load on the cover layer and strength of the cover layer, defined as expected and admissible scour, respectively) are discussed in more detail. The load and strength functions are then combined in a risk analysis. An example from engineering practice demonstrates the advantages of the critical scour method, namely that: (1) It deepens insight into the actual scour and failure processes; and (2) it incorporates additional strength parameters, such as layer thickness, which enables the risk of damage to be predicted more accurately.

INTRODUCTION

Granular cover layers are commonly used in coastal and river engineering to protect the sandy stream bed (see Fig. 1). The traditional design method is based on the "incipient motion" or "critical shear" concept. This is a practical method. However, in some cases it is not sufficient, since, for example, it is not possible to determine the effect on the stability of the bed of making the cover layer twice as thick.

For this reason, a new design philosophy has been developed based on predicting the scour occurring during the lifetime of the structure and comparing it with the admissible scour. This is called the "critical scour" method.

A summary of the critical shear method and its shortcomings is given in this paper, as well as the basic philosophy of the critical scour method. The two main parameters of this design method, load and strength, are discussed in more detail and then combined in a risk analysis. Finally, an example is given in which the critical scour method has been applied in practice.

TRADITIONAL DESIGN METHOD: CRITICAL SHEAR

The traditional design method is based on the assumption that the load can be characterized by the "maximum occurring shear stress" and the

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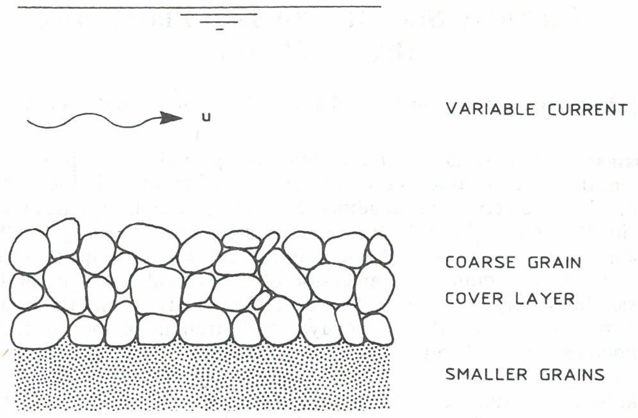

FIG. 1. Situation Considered

strength by the "critical shear stress." If expressed in a nondimensional way, the load is a function of the current velocity, the current velocity-bed shear stress relationship, and the characteristics of the grains. Load and strength can then be defined as shown in Fig. 2, where: u = current velocity; τ = bed shear stress; d = diameter of the grains; Δ = relative density of the grains; $\psi = \tau/\rho g \Delta d$ = dimensionless bed shear stress; ρ = density of water; g = acceleration due to gravity; and ψ_{cr} = dimensionless critical bed shear stress.

According to Shields (1936), the dimensionless critical bed shear stress is only a function of the Reynolds number. The uncertainties about the parameters and relationships are dealt with by applying a safety factor or risk analysis.

In this design method, it is assumed that there is no displacement of bed material if the actual bed shear stress remains below the critical value. If the bed shear stress is higher, failure is assumed. In this concept, therefore, there is a sharp boundary between no displacement and (nonadmissible) displacement, i.e., "incipient motion." In reality, this boundary is not very sharp, due to local variations in bed shear stress, grain size, and grain protrusion. [See, e.g., Paintal (1971) and Fenton and Abbott (1977)].

At bed shear stresses much lower than the critical value, according to Shields, some grain movement often occurs, whereas often at higher bed shear stresses, not all grains move simultaneously. This can be covered by defining more detailed levels of instability, varying, for example, from "displacement of some grains from time to time" to "all surface grains moving." In this way Breusers and Schukking (1971) found values of ψ_{cr} ranging from 0.03–0.07 for high Reynolds numbers. These values are based on extensive laboratory tests. The descriptions of the levels of instability are, however, of a qualitative, subjective character. It is difficult to determine what level of instability should be associated with "failure."

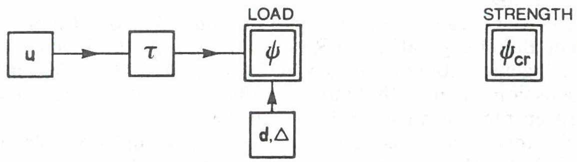

FIG. 2. Load and Strength Definitions Using Traditional Design Method

Whether a certain level of instability should be referred to as "failure" depends, among other things, on the duration of the particular state of instability. "All surface grains moving" may be acceptable, if this condition only occurs for a few seconds during the lifetime of the structure, whereas "displacement of some grains from time to time" may cause more damage if this occurs frequently.

The "critical shear" design method is fast and efficient and will remain useful in engineering practice. However, the safety factor, i.e., the ratio between the expected values of ψ_{cr} and ψ, should be rather large, e.g., 3 or 4, or, in a risk analysis, the standard deviations applied should be significant. If the minimum acceptable grain diameter is to be determined, the uncertainties about load and strength should be reduced as much as possible.

In some cases, information about currents and waves is so limited that the uncertainties associated with lack of this information are predominant. However, even if the frequency curve of the bed shear stress can be predicted accurately, there will be still a serious limitation to an optimum design because of the uncertainty about the exact value of the critical bed shear stress ψ_{cr}. Another disadvantage of the traditional design method is that the (secondary) influences of the layer thickness and the bed geometry on the strength of the bed are not taken into account.

NEW DESIGN APPROACH: CRITICAL SCOUR

The critical scour method is based on the idea that the load should be defined by the scour SC, i.e., the volume of grains eroded per unit area, and that the strength should be defined by the critical scour SC_{cr}. Fig. 3 shows this approach. The dimensionless bed shear stress ψ is found from information about currents and/or waves and grain characteristics. The transport of grains T is calculated from the shear stress, whereas the scour SC is calculated from transport and the scour length L_s. The scour length parameter is defined later. The load or scour should be compared with the critical value, influenced by the layer thickness D.

By comparing Figs. 2 and 3, it can be seen that the basic differences between the critical scour and the critical shear methods are as follows:

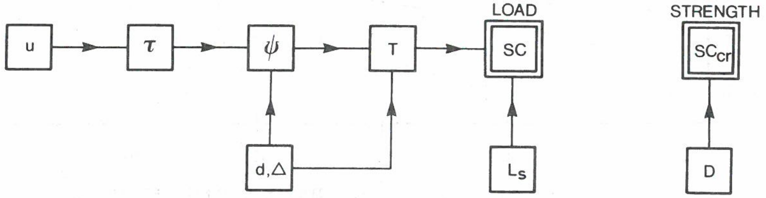

FIG. 3. Load and Strength Definitions for Critical Scour Method

1. In the critical scour method, a sharp boundary is not assumed between rest and motion. Any level of shear stress can, in fact, lead to a certain displacement of grains, i.e., transport.
2. In the critical scour method, the transport of the grains as such is not important; what is important is whether or not the transport of the grains leads to a decreasing layer thickness, i.e., scour.
3. In the critical scour method, the damage, i.e., the critical scour, should be quantified.

It should be noted that in the critical scour method scouring of the surface layer is basically accepted. This starting point might seem unsafe. However, in the traditional design method movement of the grains is inherently accepted. Even the lowest value of ψ_{cr} (ψ_{cr} = 0.03 at high Reynolds numbers) implies "displacement of some grains from time to time." Gravel transport has been measured in nature by the writers in an estuary, and it was rather surprising to find a transport of 0.015 m^3 per meter width of channel during one ebb period of six hours, although the dimensionless bed shear stress did not exceed this low value of $\psi = 0.03$ even during maximum tidal flows. In fact, these findings led to the development of the new design philosophy discussed in this paper. Designers often do not appreciate the fact that their construction may not be completely stable. The new design method requires that they quantify the expected damage.

The design method advocated here is similar to existing methods used for predicting the future level of a nonprotected sandy sea or river bed, but there are some differences, as explained hereafter.

EFFECTS OF CERTAIN PARAMETERS ON LOAD

Bed Shear Stress Frequency Distribution

The first step in determining the load is to predict the bed shear stress frequency distribution (see Fig. 4). This often requires the collection of a considerable amount of data and the application of models of the hydraulic boundary conditions (water level, currents, and waves) and the relationship between these conditions and the bed shear stress.

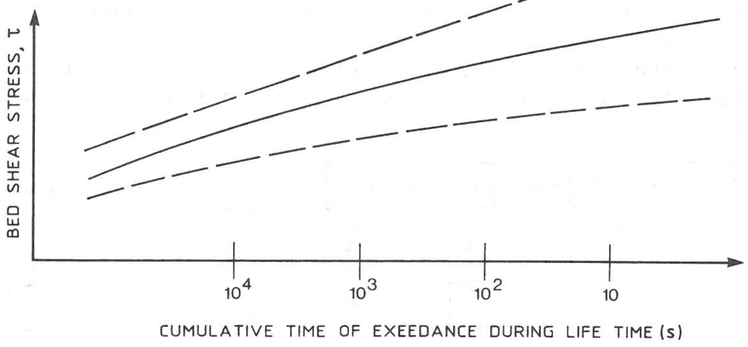

FIG. 4. Bed Shear Stress Frequency Distribution

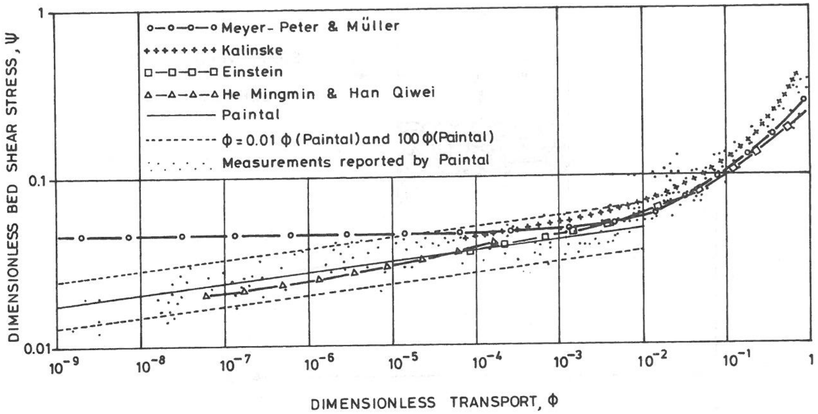

FIG. 5. Bed Load Transport at Low Shear Values

If the bed protection is, for instance, for the bed of an estuary, prediction may require knowledge of: (1) The frequency exceedance curve of the tidal range; (2) the relationship between vertical and horizontal tide; and (3) the relationship between current velocity and bed shear stress. Although relevant, it is beyond the scope of the present paper to discuss these relationships. The shear stress frequency distribution can be calculated with a certain degree of accuracy. This is shown by dotted lines in Fig. 4.

Relationship between Bed Shear Stress and Transport

An essential part of the "critical scour" method is the application of a transport formula. We are only interested in low values of bedload. Many formulas, like that of Meyer-Peter and Müller (1948), apply the concept of a fixed critical shear stress and have not been developed to give information about the very low transport levels required in the case being considered here. Einstein (1950) and Kalinske (1947), however, found ways to incorporate the stochastic character of the variables responsible for the bed load in their formulas. Others, like He and Han (1982), have also worked in this direction.

According to these theoretical formulas, some bed load transport occurs even if the average bed shear stress is very low. Extensive experiments, reported by Paintal (1971), confirm this (see Fig. 5). In this figure $\phi = T/\sqrt{\Delta g d^3}$ = dimensionless transport of grains; and T = volume of grains transported per unit width per unit time.

Fig. 5 shows the curves presented by Einstein, Kalinske, and He and Han, together with the curve fitted by Paintal into the experimental data. Paintal's formula reads

$$\phi = A \cdot \psi^B \qquad (1)$$

in which A and B = empirical constants, $A = 6.56 \times 10^{18}$, and $B = 16$. These values apply to fine material; if only coarse material ($d > 5$ mm) is

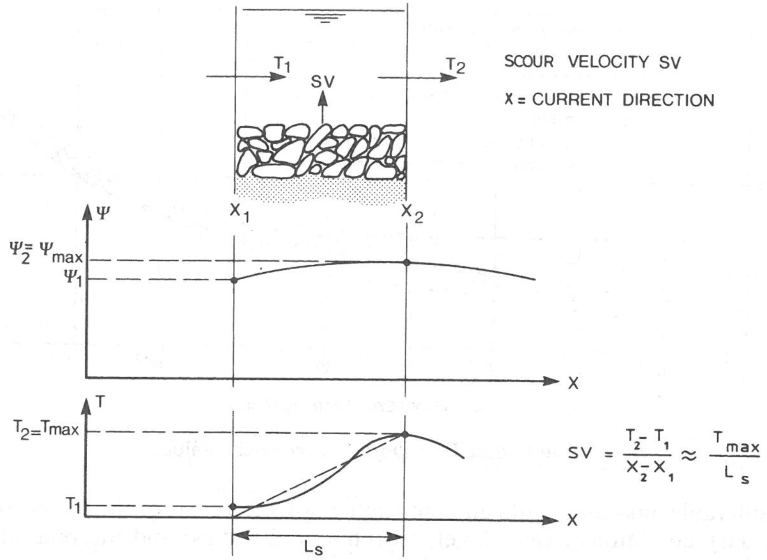

FIG. 6. Relationship between Bed Shear Stress, Transport, and Scour

used, i.e., for high Reynolds number ($\mathrm{Re}^* > 500$), the empirical constants should be adapted. According to the Delft Hydraulics Laboratory (1982), the values for coarse materials are $A = 2 \times 10^{10}$, and $B = 11$.

The simple shape of Paintal's formula is attractive for use in the critical scour method. The uncertainty about this formula and the scatter of the experimental data may be taken into account by considering A to be a stochastic variable. This is explained in the following. The dotted lines in Fig. 5 represent Paintal's formula with deviating values of A: a value 100 times as large, i.e., $A = 2 \times 10^{12}$, and a smaller value, $A = 2 \times 10^8$.

Relationship between Transport and Scour

Damage due to scour does not only depend on the transport rate. Damage only occurs if there is a horizontal transport gradient. A reasonable prediction of this gradient can be derived from only two parameters: the maximum local transport, $T_{\max}$, and the distance over which the transport rises from practically zero to the maximum value, called the "scour length" L_s (see Fig. 6). The following expression results for the scour velocity SV

$$SV = \frac{T_{\max}}{L_s} \quad \text{(2a)}$$

or, in a dimensionless form

$$\frac{SV}{\sqrt{\Delta g d}} = \frac{d}{L_s} \cdot \phi_{\max} \quad \text{(2b)}$$

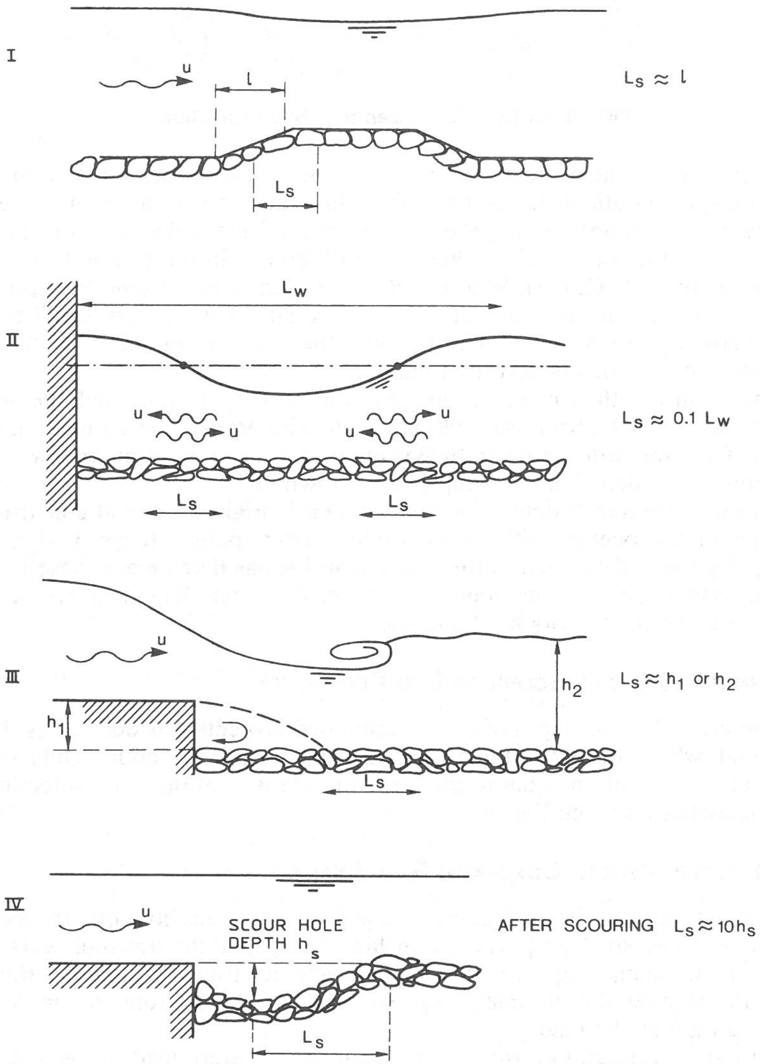

FIG. 7. Scour Length L_s in Different Circumstances

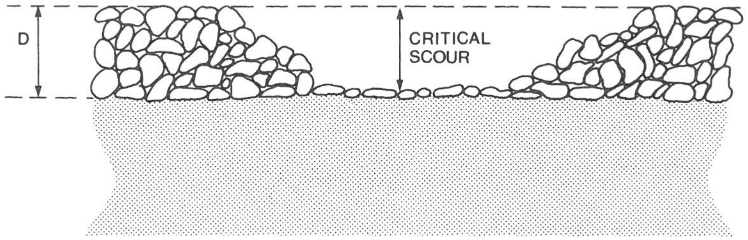

FIG. 8. Critical Scour Depth of Bed Protection

In contrast to the protected situation considered here, the calculation of the transport gradient, needed for predicting the scour of a nonprotected sandy sea or riverbed, requires knowledge of the complete transport function $T(x)$ or $\phi(x)$. This is due to the difference in the power B in the transport formula (Eq. 1). Whereas $B = 2$–3 in the case of sand transport, $B = 11$ or more in this case of coarse materials. Thus a very small bed shear stress gradient is sufficient to cause the transport of coarse material to rise from practically zero to a high value.

The scour length L_s can be estimated from the length over which the bed shear stress ψ rises from, say, 90% of its maximum value to its maximum value. Only the order of magnitude is important, as no accurate prediction of scour is needed. Some examples are shown in Fig. 7.

The scour length is determined either by the original bed and construction geometry together with the associated current pattern (cases I, II, and III) or by the bed geometry after some scouring has taken place (case IV). If, in case IV, the scour depth increases, the scour length grows, and therefore the scour velocity diminishes.

STRENGTH: CRITICAL SCOUR OF BED PROTECTION

The critical scour depth of coarse grain bed protection is defined as the point at which the underlying fine material begins to scour. Thus the critical scour is often equal to the layer thickness D of the bed protection or somewhat less (see Fig. 8).

LOAD AND STRENGTH COMBINED: RISK ANALYSIS

Combination of the bed shear stress frequency distribution, the bed shear stress versus transport relationship (Eq. 1), and the transport versus scour relationship (Eq. 2) yields a scour velocity frequency distribution. Integration over the lifetime of the structure yields the total scour, SC, defined here as the load.

With the strength, i.e., the critical scour SC_{cr}, determined, we arrive at the following requirement. For structural stability the total scour must be less than the critical scour, i.e., $SC < SC_{cr}$.

The strength and load are not known precisely, and estimates are needed. The requirement that $SC < SC_{cr}$ should be transformed to the requirement that the following probability is acceptable: $\Pr(SC > SC_{cr})$.

From Eqs. 1 and 2, and with SC_{cr} equal to the layer thickness D, this probability can be rewritten as:

$$\Pr(SC > SC_{cr}) = \Pr(D^{-1}SC > 1)$$

$$\Pr(SC > SC_{cr}) = \Pr\left(D^{-1}\int_{\text{lifetime}} SV\,dt > 1\right)$$

$$\Pr(SC > SC_{cr}) = \Pr\left(D^{-1}\int_{\text{lifetime}} L_s^{-1}\Delta^{1/2}g^{1/2}d^{3/2}\phi_{\max}\,dt > 1\right)$$

$$\Pr(SC > SC_{cr}) = \Pr\left(D^{-1}L_s^{-1}\Delta^{1/2}g^{1/2}d^{3/2}\int_{\text{lifetime}} A\psi^{B}_{\max}\,dt > 1\right)$$

$$\Pr(SC > SC_{cr}) = \Pr\left(D^{-1}L_s^{-1}\rho^{-B}\Delta^{1/2-B}g^{1/2-B}d^{3/2-B}\int_{\text{lifetime}} A\tau^{B}_{\max}\,dt > 1\right) \quad \ldots\ldots (3)$$

This probability can easily be calculated if the following assumptions are made:

1. B has a deterministic value, and all uncertainties about the transport formula can be represented by considering A to be a stochastic variable (see the dotted lines in Fig. 5).
2. $\tau_{\max}$ is a stochastic parameter for which the different values, associated with the different cumulative exceedence periods (see Fig. 4), are highly correlated to each other, e.g., if the real value of $\tau_{\max}$ is larger than the expected value of a short period, it is also larger for a long period.
3. All other parameters are independent stochastic parameters.

In view of the skewed character of the probability density functions of $\tau^{B}_{\max}$ and L_s, an easy, analytical solution can be found by assuming a log-normal distribution for all parameters.

Finally, a remark should be made about the influence of turbulence. The variation in the time over which the shear stress acts results from the variation in hydraulic boundaries (mean current velocity, wave height) and from turbulence. Paintal's formula relates transport to the turbulence-averaged shear stress in stationary uniform flow. If the turbulence is more than that found in a stationary, uniform flow, the ''excess turbulence'' might be taken into account by separating the low frequency part of the turbulence from the rest of the turbulence and adding this to the turbulence-averaged shear stress.

An alternative is to predict the frequency curve of the instantaneous shear stress. To do this, however, a transport formula must be used that, unlike Paintal's, is related to the instantaneous shear stress.

EXAMPLE OF ENGINEERING PRACTICE

The Oosterschelde storm surge barrier in the Netherlands has piers founded on a sill of densified sand. The following three construction phases are of interest here: (1) Placing the sand; (2) densifying the sand using special equipment; and (3) accurate installation of the heavy piers.

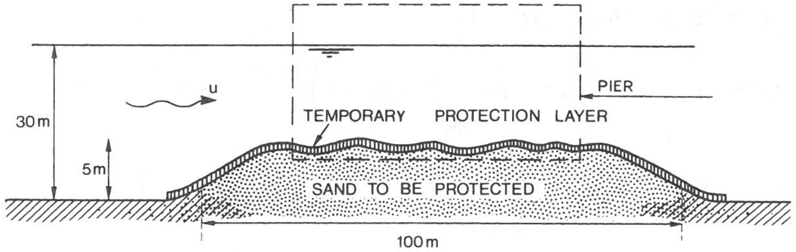

FIG. 9. Sill of Sand with Temporary Protection Layer

Due to the construction sequence, the piers could not be placed immediately after densification, and there was a period of about one year, in which scour of the costly, densified sand would create serious problems. For practical reasons, it was desirable to construct a temporary protection with a layer of fine gravel covered by gravel with a diameter not larger than 40 mm. The question was therefore raised about whether this cover layer would be sufficiently stable during the year prior to the installation of the piers (see Fig. 9).

A complication arose from the fact that the top of the sill was uneven due to the dumping process, densification, and also the sand transport before the gravel layer could be placed.

The bed could be characterized by "ripples" about 10 m long and 0.5 m high. The bed roughness therefore produced high shear stress values.

The predicted values of the parameters involved are given in Table 1, as well as the order of accuracy. The maximum or minimum values to be taken into account in view of the accuracy are assumed to correspond to the sum or difference of the predicted (expected) value E and twice the standard deviation.

The current velocity in Table 1 is the highest value to be reached in a period of 15 min during the most extreme tide expected during the lifetime of the bed protection.

From these data, the expected value of the load, the dimensionless bed shear stress ψ, was computed to be 0.02, whereas the maximum value (based on the $E \pm 2\sigma$ values) was found to be 0.12.

The expected value of the strength ψ_{cr} was taken as 0.04, whereas the minimum value was estimated to be 0.03. The resulting safety factor of $0.04/0.02 = 2.0$ is not sufficient to be able to conclude that the gravel would not be scoured away.

TABLE 1. Values of Parameters Involved

Value (1)	Current velocity, u (m/s) (2)	Chezy coefficient, C ($\mathrm{m}^{1/2}/\mathrm{s}$) (3)	Grain diameter, d (4)	Relative density, Δ (5)
Expected value, E	1.75	50	0.040	1.6
Value at $E \pm 2\sigma$	2.25	30	0.032	1.53

TABLE 2. Resulting Probability Function of Load

Scour depth, SC (m) (1)	Probability of exceedence (%) (2)
10^{-6}	74
10^{-5}	63
10^{-4}	46
10^{-3}	32
10^{-2}	18
10^{-1}	10
1	5
10	0.2

It was decided, therefore, to investigate the situation in more detail to confirm that the gravel would be acceptable. An underwater video camera was installed at a test location so that gravel displacements could be observed. Current velocity and gravel transport were measured at the location. At first sight, the transport seemed to be considerable, but whether it would cause dangerous scour was not clear. Therefore, the "critical scour" method was developed and a computer program was written.

In the program, the inaccuracy of the transport formula (Eq. 1) was represented by adopting, for the constant A, a log-normal distribution with an $A + 2\sigma$ value 100 times larger than the expected value (see Fig. 5). The resulting probability function of the load is given in Table 2.

It is interesting to consider the influence of the uncertainties of the different parameters on this result. By far the greatest contribution to the uncertainty about the scour is due to the uncertainty about the bed roughness or Chezy coefficient and the uncertainty about the current velocity. The uncertainty about the transport rate relationship, although represented by a factor of 100, appears to be less important in this example. The influence of other uncertainties, associated with, for example, scour length L_s (a factor of three), is negligible.

Combining the probability function of the load SC with the probability function of the strength SC_{cr} (expected value $= D = 0.2$ m), it was found that the bed protection had a risk of failure of nearly 10% during the period before the piers were installed. This risk was accepted, and no serious damage has been observed. If this risk had not been acceptable, the thickness D could have been increased to 0.5 m or more.

The effect of increasing the layer thickness is shown in Fig. 10. It should be noted that the scour length L_s was first estimated to be about 10% of the ripple length, i.e., $L_s = 1$ m.

The gravel transport will tend to flatten the gravel surface (no ripples will occur in the gravel at these low values of the dimensionless transport ϕ).

If the gravel layer is thin, even this small amount of transport will be sufficient to bring about failure. However, if the gravel layer is thick, the scour length L_s increases, and, therefore, the scour velocity decreases considerably before failure occurs.

In the case of the Oosterschelde, L_s becomes about 30 m, reducing the

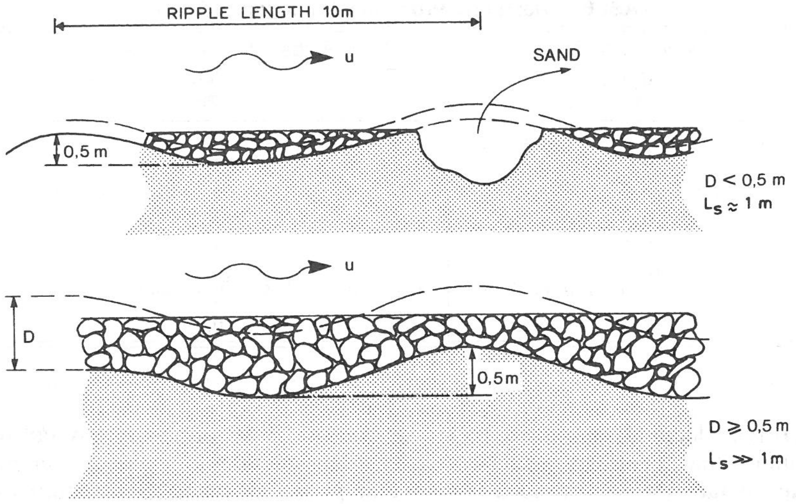

FIG. 10. Influence of Layer Thickness on Scour Length L_s

probability of failure from 10% to 1%. If the effect of a lower bed roughness, which produces a lower shear stress, is also considered, the risk of failure is reduced by another factor of ten to about 0.1%. With a nonuniform bed material, a thicker layer of gravel might also increase stability because there would be a smaller chance of size segregation.

This example of the Oosterschelde illustrates the difference between the scour of a gravel bed protection and the scour of a sandy bed. With a bed protection, the scour should be limited to less than ten times the grain diameter. In a sandy bed, scour need only be considered if it is more than, say, 1,000 times the grain diameter.

CONCLUSIONS

The application of the "critical scour" method in the design of protection of a granular bed enables the designer to learn more about the actual scour and failure processes.

Particular advantages of the critical scour method are:

1. It is not necessary to define "incipient motion" and "critical shear."
2. A reliable prediction of the actual risks of damage to the bed protection can be made that enables optimum dimensions of the cover layer material to be determined.
3. The influence of extremely high loads with very short duration and the influence of slightly lower loads with much longer duration can both be incorporated.
4. The influence of additional strength parameters, e.g., layer thickness, can be incorporated.

The example discussed in this paper demonstrates that the application of a safety factor of two, using the classical (critical shear) design method, can lead to a probability of bed protection damage of 10% in one year if only a thin (0.2 m) cover layer is applied. If a thicker cover layer (over 0.5 m) is applied, the probability of damage is reduced to about 0.1% in one year.

A disadvantage of the critical scour method, when compared with the critical shear method, is that the design calculation is relatively extensive. For this reason, the classical (critical shear) method is preferable for obtaining a first indication of bed protection stability.

More research is needed to make the critical scour method suitable for practical use. First, the predicted scour should be verified in well-controlled and relatively uncomplicated situations; a sill of uniform gravel with gentle slopes could be constructed in a flume and scoured by current flow, the details of which are known precisely. In addition, more experimental data should be collected about the relationship between the transport rate and the bed shear stress for the low transport rates, and information should be gathered concerning the relationship between local maximum bed shear stress and boundary conditions like bed geometry, current velocity, level of turbulence, and wave parameters. Finally, the influence of nonuniformity of bed material (grain size and shape) should be evaluated.

ACKNOWLEDGMENTS

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APPENDIX II. NOTATION

The following symbols are used in this paper:

A = empirical constant in Paintal's transport formula;
B = empirical constant in Paintal's transport formula;
C = Chézy coefficient ($L^{0.5}\ T^{-1}$);
D = thickness of granular bed protection layer (L);
d = grain diameter (L);
E = expected value of a stochastic parameter;
g = acceleration due to gravity (LT^{-2});
L = dimension of length;
L_s = scour length, i.e., length over which transport rises from practically zero to its maximum value (L);
M = dimension of mass;
Pr = probability (that something occurs);
Re^* = u^*d/ν = Reynolds number related to bed shear stress and grain diameter;
SC = actual scour depth (L);
SC_{cr} = critical scour depth (L);
SV = scour velocity (LT^{-1});
T = transport of grains ($L^2\ T^{-1}$); also dimension of time;
$T_{\max}$ = local maximum of T ($L^2\ T^{-1}$);
t = time (T^1);
u = actual current velocity (LT^{-1});
x = direction of current and transport (L);
Δ = $(\rho_s - \rho)/\rho$ = relative density of grains;
ν = kinematic viscosity of water ($L^2\ T^{-1}$);
ρ = density of water (ML^{-3});
ρ_s = density of grains (ML^{-3});
σ = standard deviation of stochastic parameter;
τ = actual bed shear stress ($ML^{-1}\ T^{-2}$);
$\tau_{\max}$ = local maximum of τ ($ML^{-1}\ T^{-2}$);
ϕ = $T/d\sqrt{\Delta g d}$ = dimensionless transport of grains;
$\phi_{\max}$ = local maximum of ϕ;
ψ = $\tau/\Delta\rho g d$ = actual dimensionless bed shear stress; and
ψ_{cr} = dimensionless critical bed shear stress.